Renormalization scheme for a multi-qubit-network

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We present a renormalization scheme which simplifies the dynamics of an important class of interacting multi-qubit systems. We show that a wide class of $M+1$ qubit systems can be reduced to an equivalent $n+1$ qubit system with $n \geq 2$, for *any* M. Our renormalization scheme faithfully reproduces the overall dynamics of the original system including the entanglement properties. In addition to its direct application to atom-cavity and nanostructure systems, the formalism offers insight into a variety of situations ranging from decoherence due to a spin-bath with its own internal entanglement, through to energy transfer processes in organic systems such as biological photosynthetic units.

Many-body problems are very difficult, if not impossible, to solve exactly. Any simplifications are therefore of great potential importance – not only because of practical applications but also because of a basic theoretical interest. Such simplifications usually result if some exact or approximate symmetry can be identified in the underlying Hamiltonian (e.g. Ref. [1]). In terms of practical applications, most exact results to date have concerned systems where all the interacting objects (e.g. particles) are indistinguishable (e.g. Ref. [2]). This is understandable, since systems such as many-electron gases have been of great experimental interest over the past few decades. However given the current levels of activity in the field of quantum information processing, there is a clear desire to develop such theoretical results for multi-qubit systems. In particular, it would be highly desirable to obtain exact or approximate results for the generic situation in which a collection of qubits interacts with an auxiliary system such as a cavity mode [3] or a central spin [4].

Here we show that a system of $M+1$ qubits whose interactions resemble a spin-star configuration (see Fig. 1) can be mapped onto an equivalent $n+1$ interacting system with $n \geq 2$, preserving the dynamics of the central qubit *and* the quantum correlations of the original system. This setup can be realized using multi-atom-cavity systems, and could also be engineered from a collection of quantum dots in an optical cavity. It also mimics naturally-occuring photosynthetic complexes (see Fig. 2) which are of fundamental importance in nature. Moreover, we will show that it yields insight into the decoherence properties of spin-baths possessing internal entanglement.

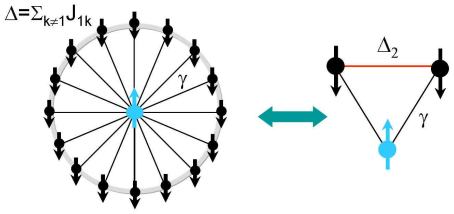

FIG. 1: Schematic representation of the renormalization scheme in which an interacting $M+1$ multi-qubit with one excitation can be reduced to an equivalent $n+1$ qubit system with one excitation, where $n \geq 2$. For example, the excitation can correspond to having a central qubit in its excited state (spin up) while the outer qubits are in their ground state (spin down). A wide range of interactions are possible between outer qubits, including nearest-neighbor, dipole-dipole, and pairwise interactions between any pair in the ring.

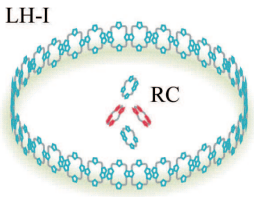

FIG. 2: Schematic diagram of the light-harvesting complex LH-I and reaction centre RC structure in purple bacteria. For details see Ref. [8]. This photosynthetic unit closely resembles our model system in Fig. 1.

The key physical feature which underpins this equivalence, is a particular non-trivial symmetry in the interactions experienced by all outer qubits. This symmetry allows us to describe the system's dynamics in terms of two time-dependent variables, associated with the outer qubits and central qubit (or cavity mode) respectively. We consider M identical qubits which interact among themselves and with a central qubit via the following interaction Hamiltonian (c.f. Fig. 1):

$$H_I = \sum_{j=1}^{M} \gamma_{jC}\{\sigma_C^+\sigma_j^- + \sigma_j^+\sigma_C^-\} + \sum_j \sum_{k\neq j} J_{jk}\{\sigma_j^+\sigma_k^- + \sigma_k^+\sigma_j^-\} \quad (1)$$

where $\sigma_j^{+(-)}$ is the usual raising (lowering) operator for j'th outer qubit, or the central qubit with $j \equiv C$. The formalism and results in this paper apply to a wide range of possible two-body interactions J_{jk}, e.g. short-range (i.e. nearest-neighbor), long-range (i.e. pairwise,

between any pair in the ring) and dipole-dipole. The Hamiltonian H_I preserves the number of excitations, i.e $[H_I, \mathcal{N}] = 0$ with $\mathcal{N} = \sum_{j=1}^{M} \sigma_j^+ \sigma_j^- + \sigma_C^+ \sigma_C^-$. We focus first on the dynamics within the single-excitation subspace. A basis is given by states in which one qubit is excited and the rest are in their ground state, i.e. $\{|q_1\rangle|0_C\rangle, |q_j\rangle|0_C\rangle, ...|q_M\rangle|0_C\rangle, |0_B\rangle|1_C\rangle\}$ with

$$\begin{aligned} |q_j\rangle &= |0_1, 0_2 \cdots 1_j \cdots 0_M\rangle \\ |0_B\rangle &= |0_1, 0_2 \cdots 0_j \cdots 0_M\rangle \end{aligned} \tag{2}$$

with $j = 1, \ldots M$. The state of this M-plus-central qubit system is given by the unitary evolution associated with H_I:

$$|\Psi(t)\rangle = |1_B\rangle|0_C\rangle + b_C(t)|0_B\rangle|1_C\rangle \tag{3}$$

with $|1_B\rangle = \sum_{j=1}^{M} b_j(t)|q_j\rangle$ and $\sum_{j=1}^{M} |b_j(t)|^2 + |b_C(t)|^2 = 1$. We have written the state in this way in order to emphasize the collective behaviour of the outer qubits – indeed, as a reference to one of the potentially important applications of this work, we will frequently use the term *bath* to denote this collection of outer qubits. The system's state satisfies the Schrödinger equation $d|\Psi(t)\rangle/dt = -iH_I|\Psi(t)\rangle$, which leads to a set of first-order coupled differential equations for the complex amplitudes $b_j(t)$ and $b_C(t)$. Due to a collective symmetry in the effective interaction experienced by all outer qubits, it is possible to describe the system's dynamics in terms of two time-dependent variables $b_c(t)$ and $B(t) = \sum_{j=1}^{M} b_j(t)$ associated with the central qubit and all outer qubits respectively. The nature of this symmetry is as follows: Let us assume that each one of the outer qubits interacts with the center via an identical coupling $\gamma_{jC} \equiv \gamma$. Because of this, the strength of the effective interaction between a qubit j and the rest of the outer qubits is captured by a parameter $\Delta_j = \sum_{k \neq j} J_{jk}$, i.e. the sum of all the coupling strengths between qubit j and any other outer qubit. When Δ_j is identical for all qubits, i.e. $\Delta_j \equiv \Delta$, as it is for example in the case of nearest-neighbour or pairwise interactions, the system's dynamics can be described in terms of $b_c(t)$ and $B(t)$. Note that this does not imply that the pair couplings J_{jk} need to be identical for all possible pairs. Instead it implies that the collective interaction experienced by one of the outer qubits, due to the rest, should be identical for all qubits. Hence the symmetry is such that the interaction experienced for each qubit (central or outer) is of a collective nature. Given this symmetry, $b_c(t)$ and $B(t)$ satisfy the following set of differential equations: $\ddot{b}_C(t) + i\Delta\dot{b}_C(t) + 4M\gamma^2 b_C(t) = 0$ and $\dot{B}(t) + i\Delta B(t) = -iM\gamma b_C(t)$, with solutions of the form $b_C(t) = f_C(t)B(0) + g_C(t)b_C(0)$ and $B(t) = f_B(t)B(0) + g_B(t)b_C(0)$ where

$$\begin{aligned} f_C(t) &= -ie^{-i\Delta t/2} 2\gamma u(t)/\Omega \\ g_C(t) &= e^{-i\Delta t/2}[(i\Delta/\Omega)u(t) + v(t)] \\ f_B(t) &= e^{-i\Delta t/2}[(i\Delta/\Omega)u(t) - v(t)] \\ g_B(t) &= -ie^{-i\Delta t/2} 2M\gamma u(t)/\Omega \end{aligned} \tag{4}$$

with $u(t) = \sin(\Omega t/2)$, $v(t) = \cos(\Omega t/2)$, and

$$\Omega = \Omega_M(\gamma, \Delta) = \sqrt{4M\gamma^2 + \Delta^2} \tag{5}$$

is a frequency that captures the collective character of the interactions in the system. Note that $2\sqrt{M}\gamma$ captures the collective features of the interaction between the outer and central qubits, while Δ captures the collective interaction between one outer qubit and the remaining $M-1$ qubits. The relevant expectation values for the dynamics of the central qubit are determined by the reduced density operator $\rho_C(t) = \mathrm{Tr}_{outer}\{|\Psi(t)\rangle\langle\Psi(t)|\}$. For an initial state of the same form as Eq.(3) we have

$$\rho_C(t) = |b_C(t)|^2 |0_C\rangle\langle 0_C| + (1 - |b_C(t)|^2)|1_C\rangle\langle 1_C| \ . \tag{6}$$

Hence all the physical properties of the central qubit are determined by $b_C(t)$ and the relevant expectation values can be calculated, e.g. $\langle \sigma_c^z \rangle = 2|b_C(t)|^2 - 1$.

The system's dynamics are characterized by two effective interaction strengths γ and Δ. This suggests that the entanglement properties should also be describable in terms of two contributions [5]: one corresponding to the entanglement among all outer qubits, which we call $\mathcal{E}_B$, and another corresponding to the entanglement between the central qubit and the rest, which we call $\mathcal{E}_{BC}$. The time-dependent versions of these quantities are given by

$$\mathcal{E}_B(t) = |-1 + |B(t)|^2 + |b_C(t)|^2| \tag{7}$$

and

$$\mathcal{E}_{BC}(t) = 4|b_C(t)|^2(1 - |b_C(t)|^2) \tag{8}$$

We now take advantage of some known results for $W-$states since these are the states of interest in our multi-qubit system (see Eq. 3). In particular, for any partition in the system the entanglement is entirely composed of pairwise contributions [5] that can be quantified by the concurrence [9]. The concurrence of the reduced state of two qubits in our system has the form $C_{jk} = 2|b_j(t)b_k(t)^*| \leq 1$. Hence a measure of the total entanglement among the outer qubits is $E_B = \sum_{\langle j,k \rangle} C_{jk}$, while the total entanglement between the central and outer qubits is $E_{BC} = \sum_{j=1}^{M} C_{jC}$. We now demonstrate that $\mathcal{E}_B(t)$ and $\mathcal{E}_{BC}(t)$ are lower bounds of E_B and E_{BC}, respectively i.e. $\mathcal{E}_B(t) \leq E_B$, and $\mathcal{E}_{BC}(t) \leq E_{BC}$, and hence they can be used to quantify the entanglement properties of our system. We express the complex amplitudes $b_j(t)$ as $b_j(t) = |b_j(t)|e^{i\theta_k}$ hence

$$\begin{aligned} |B(t)|^2 &= \sum_{j=1}^{M}\sum_{k=1}^{M} |b_j(t)b_k(t)^*| e^{i\alpha_{jk}} \ \text{ with } \alpha_{jk} = \theta_j - \theta_k \\ &= \sum_{j=1}^{M} |b_j(t)|^2 + \sum_{\langle j,k \rangle} C_{jk}\cos(\alpha_{jk}) \end{aligned}$$

Using $1 = \sum_{j=1}^{M} |b_j(t)|^2 + |b_C(t)|^2$, we have

$$\mathcal{E}_B(t) = \left| \sum_{\langle j,k \rangle} C_{jk} \cos(\alpha_{jk}) \right| \leq \sum_{\langle j,k \rangle} C_{jk} |\cos(\alpha_{jk})| \leq \sum_{\langle j,k \rangle} C_{jk}$$

Knowing that $C_{jC} \leq 1$, we then have that $E_{BC} \geq \sum_{j=1}^{M} C_{jC}^2 = 4|b_C(t)|^2(1-|b_C(t)|^2) = \mathcal{E}_{BC}(t)$ which completes our proof.

An important observation from the above relations is that the central qubit dynamics (Eq.(6)) and the central-outer qubit entanglement (Eq. (8)) are both completely determined by the probability of having the excitation on the central qubit, i.e. $|b_C(t)|^2$, while the intra-bath entanglement $\mathcal{E}_B(t)$ depends on both $|b_C(t)|^2$ and $|B_0(t)|^2$. Therefore $|b_C(t)|^2$ and $\mathcal{E}_B(t)$ are the relevant quantities to characterize the dynamics of the $M+1$-qubit system. Henceforth we shall refer to the quantities as $P_M(t) = |b_C(t)|^2$ and $E_M(t) = \mathcal{E}_B(t)$ where the subscript M indicates that these quantities correspond to the $M+1$ system.

The above considerations lead to the following dynamical equivalence, which we shall now discuss: *A system of $M+1$ qubits with interactions forming a spin-star configuration and characterized by two parameters γ and Δ, is dynamically equivalent to a system of $n+1$ qubits in a similar configuration and characterized by Δ_n and γ_n* given that their collective frequencies are identical $4M\gamma^2+\Delta^2 = 4n\gamma_n^2+\Delta_n^2$. Since γ and Δ are independent of each other, we can fix γ in the equivalent and original systems, i.e $\gamma = \gamma_n$, such that

$$\Delta_n^2 = 4(M-n)\gamma + \Delta^2 \tag{9}$$

The above statements imply that we can find transformations between the dynamical quantities of the original and equivalent systems, i.e. $P_M(t) = \mathcal{F}(P_n(t))$ and $E_M(t) = \mathcal{F}(E_n(t))$. These transformations are:

$$\begin{aligned} \frac{P_M(t) - P_M(0)}{\alpha_M} &= \frac{P_n(t) - P_n(0)}{\alpha_n} \\ \frac{E_M(t) - E_M(0)}{\beta_M} &= \frac{E_n(t) - E_n(0)}{\beta_n} \end{aligned} \tag{10}$$

with $\alpha_n = |B_n(0)|^2 - nP_n(0)$ and $\beta_n = n(n-1)P_n(0) + (n-1)|B_n(0)|^2$. Notice that β_n sets the minimum n in the above relations to two, i.e. $n_{min} = 2$. Therefore *in order to preserve the entanglement properties of the original system, the equivalent system should have at least two outer qubits.* Hence as far as the dynamics of the central qubit and the overall entanglement properties are concerned, it is possible to map an $M+1$ qubit system onto a system of $2+1$ interacting qubits (c.f. Fig. 1). As an example, consider the case where the excitation is initially on the central spin. The time-evolution for $P_M(t)/M$ and $E_M(t)$ in an original system with $M = 10$ and the equivalent system $M = 2$, are shown in Fig. 2.

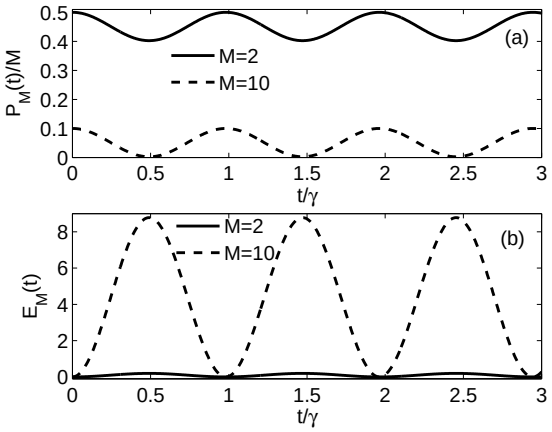

FIG. 3: (a) Dynamics of central qubit, and (b) intra-bath entanglement properties for original and equivalent systems when the excitation is initially on the central qubit, i.e. $b_c(0) = 1$. Nearest-neighbour interactions have been assumed such that $\Delta = J$ and $\Omega = \sqrt{40\gamma + J}$. We take $J = \gamma$.

They are described by the following relations:

$$\begin{aligned} P_M(t) &= \left[1 - \frac{M}{2}\right] + \frac{M}{2} P_2(t) \\ E_M(t) &= \frac{M(M-1)}{2} \mathcal{E}_2(t) \ . \end{aligned} \tag{11}$$

We now discuss the effect of dissipation in the system using the quantum jump approach[6]. We assume identical decay-rates for the outer qubits Γ, but these can be different from the decay-rate κ of the central qubit. The non-unitary dynamics conditioned on 'no-loss' of excitation is given by $H_{cond} = H_I - i\Gamma \sum_{j=1,M} \sigma_j^+ \sigma_j^- + i\kappa\sigma_C^+\sigma_C^-$. The unnormalized state of the system is $|\Psi_{cond}(t)\rangle = \exp[-iH_{cond}t]|\Psi(0)\rangle$ which can be expressed in the same form as in Eq.(3). Hence, we can again describe the system's dynamics in terms of (normalized versions of) $b_C(t)$ and $B(t)$. In this case, one obtains:

$$\begin{aligned} f_C(t) &= -ie^{-Xt/2} 2\gamma u(t)/\Omega \\ g_C(t) &= e^{-Xt/2} \left[\frac{(\delta + i\Delta)}{\Omega} u(t) + v(t) \right] \\ f_B(t) &= e^{-Xt/2} \left[\frac{(\delta + i\Delta)}{\Omega} u(t) - v(t) \right] \\ g_B(t) &= -ie^{-Xt/2} 2M\gamma u(t)/\Omega \end{aligned} \tag{12}$$

where $X = \kappa + \Gamma + i\Delta$, $\delta = \Gamma - \kappa$ is the effective dissipation coefficient, and $u(t)$ and $v(t)$ are defined as before. The collective frequency is now given by

$$\Omega = \Omega_M(\gamma, \Delta, \delta) = \sqrt{4M\gamma^2 - (\delta + i\Delta)^2} \ . \tag{13}$$

With the above equations, one can obtain a similar renormalization scheme to the one discussed earlier.

The above formalism can be immediately applied to model a quantum spin-bath with intra-environmental

coupling. The spin-bath model has typically been employed in the literature to describe the system-bath interaction at low temperatures. In order to achieve tractable approaches, interactions among the bath spins have usually been neglected. However, recent works have started to explore whether the interaction among bath spins might indeed have significant knock-on effects on the dynamics of the central qubit [5, 7]. These studies suggest that the intra-bath interactions can suppress decoherence of the central spin. It has also been argued that such an effect is due to the fact that the intra-bath entanglement limits the spin-bath entanglement, and hence the decoherence of the central spin [5]. These features can be easily understood within our approach, as follows. The intra-bath interaction is represented by the effective coupling Δ, which affects the dynamics of the central spin through the collective effective frequency Ω. The relation between the spin-bath entanglement ($\mathcal{E}_{BC}(t)$) and the intra-bath entanglement ($\mathcal{E}_B(t)$) can be seen explicitly in Eq.(7) where it is clear that $\mathcal{E}_B(t)$ limits $|b_C(t)|^2$, thereby limiting the spin-bath entanglement. Most importantly, our approach indicates that such complex many-body features can be simulated by a simple system of only *three* interacting qubits.

Our theoretical formalism can also be used to investigate the excitation transfer between a light-harvesting LH-I complex and the reaction centre RC in photosynthetic bacteria (see Fig. 2 and Ref. [8]). The LH-I ring is made up of 32 donor units – each of these [8], in addition to the RC, can be treated as a two-level system to a good approximation. The prevalent interaction between donors is given by an induced dipole-dipole coupling which can be approximated as $J_{jk} \simeq J/r_{jk}^3$ where r_{jk} is the relative position vector between the outer qubits j and k. To a good approximation, all the induced dipole moments can be taken as identical and lying perpendicular to the plane containing the outer qubits, yielding $\Delta = J\sum_{k=2}^{M}(1/r_{1k})^3$. Hence, as far as the dynamics of the center is concerned, this complicated 32-donor ring can be accurately represented by *two* donors. We do not pursue this any further here, but leave it as an interesting consequence of the present theoretical study.

We now discuss the extension of these results to larger numbers of excitations in the multi-qubit system. Since the central qubit behaves as a spin-$\frac{1}{2}$ particle, only two dimensions of the bath's state-space are required to expand the pure state in its Schmidt decomposition. For any number of excitations $N \leq M$, we can express the system's state in an analogous way to Eq.(3)

$$|\Psi(t)\rangle = |N_B\rangle|0_C\rangle + |N-1_B\rangle|1_C\rangle \qquad (14)$$

where $|A_B\rangle$ is a time-dependent superposition of $\binom{M}{A}$ states, each one having A excitations. We can therefore justifiably claim that the dynamics of a system with $N \leq M$ excitations is analogous to the case of $M+1-N$ excitations. A particular case is $N = M$ whose solutions are analogous to the case of single excitation case we have discussed. This can be seen in Fig. 1 but now interpreting the presence of an excitation as a spin-down. Therefore, *a system with $M+1$ qubits and M excitations can be mapped on to a $2+1$ system with 2 excitations.* The complexity of the dynamics is, however, highly nontrivial for $M-N>0$. Although we don't currently have a full solution, insight into this problem can be gained by analyzing the effect of H_I on $|\Psi(t)\rangle$ in detail. In particular, we re-write $H_I = V_B + V_{BC}$ where V_B is the interaction between central qubit and the outer qubits, and V_{BC} represents the interaction among outer qubits. V_{BC} induces transitions from $|N_B\rangle|0_C\rangle$ to $|N-1_B\rangle|1_C\rangle$ while V_B just produces internal transitions in each $|A_B\rangle$, since it preserves the internal number of excitations in the outer qubits. Hence we conjecture that for any $N \leq M$, the dynamics can be described in terms of two collective variables: one associated with $|N_B\rangle|0_C\rangle$ and the other with $|N-1_B\rangle|1_C\rangle$. The open question for future study then becomes: What is the *minimum* number of qubits needed to represent such collective properties?

This paper has presented new equivalence relations involving multi-qubit systems. In particular, this formalism provides a new way of simplifying the system-bath interactions in open quantum systems.

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